Polarization Method for Static Fields

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***Abstract*—An overview of the polarization method is presented. The method can by applied for different regimes of the electromagnetic field as well as for electric circuits. Criteria for the choice of the permeability are proposed, so that the iterative scheme leads to a Picard–Banach fixed point procedure. The errors are evaluated. An efficient overelaxation method is presented. The modality of using FEM numerical method is analyzed in order to ensure the convergence of the method.**

***Index Terms*—Finite elements, fixed point, magnetic field, nonlinear media.**

I. Introduction

TWO IMPORTANT iterative schemes can be used for solving the nonlinear equation $\hat{\boldsymbol{F}}(\boldsymbol{x}) = 0$. The simplest one seems to be the Picard–Banach scheme that leads to the sequence: $\boldsymbol{x}^{(n+1)} = \boldsymbol{x}^{(n)} - \mu\hat{\boldsymbol{F}}(\boldsymbol{x}^{(n)}) = \hat{\boldsymbol{G}}(\boldsymbol{x}^{(n)})$. If the function $\hat{\boldsymbol{F}}$ is Lipschitzian and uniform monotone, then the function $\hat{\boldsymbol{G}}$ is a contraction and the Picard–Banach sequence leads to the fixed point of the function $\hat{\boldsymbol{G}}$ that is also the solution of the equation $\hat{\boldsymbol{F}}(\boldsymbol{x}) = 0$. This procedure was proposed for the first time, in electrical engineering, by Katzenelson and Seitelman [1] for nodal analysis of resistive electric circuits. Unfortunately, the method proposed in [1] did not have success because the contraction factor is very close to unit and the convergence of the iterative procedure is very slow. The most used scheme for solving the equation $\hat{\boldsymbol{F}}(\boldsymbol{x}) = 0$ seems to be Newton–Raphson method that leads to the sequence: $\boldsymbol{x}^{(n+1)} = \boldsymbol{x}^{(n)} - [(d\hat{\boldsymbol{F}}/d\boldsymbol{x}^{(n)})]^{-1}\hat{\boldsymbol{F}}(\boldsymbol{x}^{(n)})$. The conditions that ensure the convergence of the Newton–Raphson sequence imply bounds for $[(d\hat{\boldsymbol{F}}/d\boldsymbol{x})]^{-1}$ and $[(d^2\hat{\boldsymbol{F}}/d\boldsymbol{x}^2)]$. Almost always they can not be emphasized. Very often the Newton–Raphson method must be supplemented by underrelaxation procedures to avoid troubles. However we must admit that the Newton–Raphson method is the most efficient one for solving the equation $\hat{\boldsymbol{F}}(\boldsymbol{x}) = 0$, when it is convergent.

The Picard–Banach fixed point procedure was proposed for solving magnetic field problems in 1975 [2]. The magnetic polarization $\boldsymbol{I}$ is nonlinear related to the magnetic flux density $\boldsymbol{B}$. The magnetic field is solved in a linearized medium for which $\boldsymbol{B} = \mu\boldsymbol{H} + \boldsymbol{I}$. By choosing an optimum permeability such that the smallest contraction factor will result [3] and an overrelaxation procedure [4], the convergence of the polarization method is significantly improved. Important advantages result in comparison with the Newton–Raphson method. The method can be applied also for nonlinear $\boldsymbol{E}$–$\boldsymbol{J}$ constitutive relations and for electric circuits having nonlinear resistors. In hysteretic material media, where the Newton–Raphson scheme can not be used, the polarization method is applicable [5].

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II. Equations of the Magnetic Field

Let Ω be a domain and $\partial\Omega$ its boundary. In Ω the magnetic field verifies the following equations:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} \tag{1}$$

$$\nabla \boldsymbol{B} = 0 \tag{2}$$

$$\boldsymbol{H} = \hat{\boldsymbol{F}}(\boldsymbol{B}) \tag{3}$$

Non-linear function $\hat{\boldsymbol{F}} : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ refers to the domains with ferromagnetic bodies and permanent magnets. Usually the relation $\boldsymbol{B}$–$\boldsymbol{H}$ is locally defined in almost all points P of the domain Ω.

$$\boldsymbol{H}(P) = \boldsymbol{f}(P, \boldsymbol{B}(P)) \tag{4}$$

If the function $\boldsymbol{f}$ is Lipschitzian:

$$|\boldsymbol{f}(\boldsymbol{B}') - \boldsymbol{f}(\boldsymbol{B}'')| < \Lambda(P)|\boldsymbol{B}' - \boldsymbol{B}''|, (\forall)\boldsymbol{B}', \boldsymbol{B}''$$

and uniform monotone:

$$(\boldsymbol{f}(\boldsymbol{B}') - \boldsymbol{f}(\boldsymbol{B}''))(\boldsymbol{B}' - \boldsymbol{B}'') > \lambda(P)(\boldsymbol{B}' - \boldsymbol{B}'')^2, (\forall)\boldsymbol{B}', \boldsymbol{B}''$$

where $\Lambda(P) < \Lambda_M$ and $\lambda(P) > \lambda_m > 0$ in Ω, then the function $\hat{\boldsymbol{F}}$ is Lipschitzian and uniform monotone. For example, in an isotropic medium we have:

$$\Lambda(P) = \sup_{\boldsymbol{B}', \boldsymbol{B}''} \frac{|\boldsymbol{f}(\boldsymbol{B}') - \boldsymbol{f}(\boldsymbol{B}'')|}{|\boldsymbol{B}' - \boldsymbol{B}''|} = \nu_{\max};$$

$$\lambda(P) = \inf_{\boldsymbol{B}', \boldsymbol{B}''} \frac{|\boldsymbol{f}(\boldsymbol{B}') - f(\boldsymbol{B}'')|}{|\boldsymbol{B}' - \boldsymbol{B}''|} = \nu_{\min}$$

III. The Polarization Method

Relation (3) is replaced by:

$$\boldsymbol{B} = \mu\boldsymbol{H} + \boldsymbol{I} \tag{5}$$

where the nonlinearity is hidden [4] in the polarization $\boldsymbol{I}$:

$$\boldsymbol{I} = \boldsymbol{B} - \mu\hat{\boldsymbol{F}}(\boldsymbol{B}) = \hat{\boldsymbol{G}}(\boldsymbol{B}) \tag{6}$$

The permeability μ may be chosen so that the function $\hat{\boldsymbol{G}}$ defined by relation (6) is a contraction:

$$\|\hat{\boldsymbol{G}}(\boldsymbol{B}') - \hat{\boldsymbol{G}}(\boldsymbol{B}'')\|_\nu \le \theta\|\boldsymbol{B}'\text{-}\boldsymbol{B}''\|_\nu, (\forall)\boldsymbol{B}', \boldsymbol{B}''$$

where $\theta < 1$, $\langle \boldsymbol{X}, \boldsymbol{Y}\rangle_\nu = \langle \boldsymbol{X}, \ \nu\boldsymbol{Y}\rangle = \int_\Omega \boldsymbol{X}\nu\boldsymbol{Y}\, d\Omega$ and $\nu = 1/\mu$. For $\mu \in (0, 2\lambda_m/\Lambda_M^2)$, we have $\theta \le 1 - 2\mu\lambda_m + \mu^2\Lambda_M^2 < 1$. The smallest value of the contraction factor θ is

$1-(\lambda_m/\Lambda_M)^2$ and it is obtained for $\mu=\lambda_m/\Lambda_M^2$. This contraction factor is very close to unit (consider, for example, a ferromagnetic medium having $\Lambda=1/\mu_0$ and $\lambda=1/1000\mu_0$ for which the smallest contraction factor is 0.999 999). Other procedures for choosing the permeability μ are more convenient [4]. In the case of an isotropic media we can choose in any point $\mu(P)<2\mu_{\min}(P)$, so that the function $\hat{\boldsymbol{G}}$ is a contraction. The contraction factor θ is:

$$\theta=\operatorname*{Sup}_{P\in\Omega}[\operatorname{Max}(1-\mu(P)/\mu_{\max}(P),\,\mu(P)/\mu_{\min}(P)-1)].$$

Since $\mu_{\min}\geq\mu_0$, we can replace the nonlinear medium by a linear one having the permeability of the vacuum. In this case $\theta=1-\mu_0/\mu_M$, where $\mu_M=\operatorname{Sup}_{P\in\Omega}\mu_{\max}(P)$. The smallest value of θ is $\operatorname{Sup}_{P\in\Omega}(\mu_{\max}(P)-\mu_{\min}(P)/\mu_{\max}(P)+\mu_{\min}(P))$ and it is obtained for $\nu_{opt}(P)=(\nu_{\min}(P)+\nu_{\max}(P))/2$.

A. The Iterative Method

The algorithm is as follows:

a) *An arbitrary value $I^{(0)}$ is given*;

b) *The magnetic field $(\boldsymbol{B}^{(n)},\boldsymbol{H}^{(n)})$ which verifies equations*: $\nabla\times\boldsymbol{H}^{(n)}=\boldsymbol{J}$, $\nabla\boldsymbol{B}^{(n)}=0$, $\boldsymbol{B}^{(n)}=\mu\boldsymbol{H}^{(n)}+\boldsymbol{I}^{(n-1)}$ *is computed*;

c) $\boldsymbol{I}^{(n)}$ *is corrected with relation (6)*: $\boldsymbol{I}^{(n)}=\hat{\boldsymbol{G}}(\boldsymbol{B}^{(n)})$; *Step b) and c) are repeated until*:

$$\left\|\Delta\boldsymbol{I}^{(n)}\right\|_\nu=\left\|\boldsymbol{I}^{(n)}-\boldsymbol{I}^{(n-1)}\right\|_\nu=\sqrt{\int_\Omega\nu\left(\boldsymbol{I}^{(n)}-\boldsymbol{I}^{(n-1)}\right)^2d\Omega}<\varepsilon,$$

where ε is an imposed error.

For any $\boldsymbol{I}$ we have one and only one field $\boldsymbol{B}$ who verifies (1), (2) and (5) [6]. The function $\boldsymbol{I}\rightarrow\boldsymbol{B}=\hat{\boldsymbol{B}}(\boldsymbol{I})$ is nonexpansive: $\|\hat{\boldsymbol{B}}(\boldsymbol{I}')-\hat{\boldsymbol{B}}(\boldsymbol{I}'')\|_\nu\leq\|\boldsymbol{I}'-\boldsymbol{I}''\|_\nu$. Indeed, let $(\boldsymbol{B}',\boldsymbol{H}')$, $(\boldsymbol{B}'',\boldsymbol{H}'')$ be two fields having the same boundary conditions, the same current densities $\boldsymbol{J}$ and polarization $\boldsymbol{I}'$ and $\boldsymbol{I}''$, respectively. The difference $(\boldsymbol{B}_d,\boldsymbol{H}_d)$ of these fields verifies relation $\nabla\times\boldsymbol{H}_d=0$, $\nabla\boldsymbol{B}=0$ and satisfies null boundary conditions. Therefore it verifies the relation: $\langle\boldsymbol{B}_d,\boldsymbol{H}_d\rangle=0$. Taking into account the relation $\boldsymbol{H}_d=\nu(\boldsymbol{B}_d-\boldsymbol{I}_d)$ it results: $\|\boldsymbol{B}_d\|_\nu^2=\langle\boldsymbol{B}_d,\boldsymbol{I}_d\rangle_\nu$. Hence, $\|\boldsymbol{B}_d\|_\nu^2\leq\|\boldsymbol{I}_d\|_\nu^2$. Because the function $\hat{\boldsymbol{G}}$ is contractive, the above iteration scheme is a Picard–Banach fixed point procedure to find the fixed point of the function $\hat{\boldsymbol{W}}=\hat{\boldsymbol{G}}\circ\hat{\boldsymbol{B}}$.

Theorem 1: μ may be chosen so that the function $\hat{\boldsymbol{G}}$ is a contraction and then the Polarization Method using B-correction is convergent.

B. The H-Correction

A dual formulation may be used for the treatment of the nonlinearity. Instead of relations (5) and (6), we have: $\boldsymbol{H}=\nu\boldsymbol{B}-\boldsymbol{M}$. Here the nonlinearity is hidden in the magnetization term $\boldsymbol{M}$: $\boldsymbol{M}=\boldsymbol{H}-\mu\hat{\boldsymbol{F}}^{-1}(\boldsymbol{H})=\hat{\boldsymbol{G}}'(\boldsymbol{H})$.

We can choose ν so that $\hat{\boldsymbol{G}}'$ is a contraction. For example, in the case of an isotropic media we have $\nu<2\nu_{\min}$, namely $\mu>\mu_{\max}/2$. The function $\boldsymbol{M}\rightarrow\boldsymbol{H}=\hat{\boldsymbol{H}}(\boldsymbol{M})$ is nonexpansive

$$\left\|\hat{\boldsymbol{H}}(\boldsymbol{M}')-\hat{\boldsymbol{H}}(\boldsymbol{M}'')\right\|_\mu\leq\left\|\boldsymbol{M}'-\boldsymbol{M}''\right\|_\mu$$

The iterative scheme corrects the magnetization $\boldsymbol{M}$ as an $\boldsymbol{H}$-function. It is also a Picard–Banach fixed point procedure to find the fixed point of the function $\hat{\boldsymbol{W}}'=\hat{\boldsymbol{G}}'\circ\hat{\boldsymbol{H}}$. Therefore:

Theorem 2: μ can be chosen so that the function $\hat{\boldsymbol{G}}'$ is a contraction and then the Polarization Method using $\boldsymbol{H}$-correction is convergent.

Note. The choice $\mu=\mu_0$ does not ensure the contractivity of the function $\hat{\boldsymbol{G}}'$ and the convergence of the $\boldsymbol{H}$-correction iterative method.

A hybrid scheme may be used: $\boldsymbol{H}$-correction is done in a region Ω_H and $\boldsymbol{B}$-correction is performed in the rest Ω_B

C. Errors

The error in comparison with the exact solution $(\boldsymbol{B}^*,\boldsymbol{H}^*)$ can be easily evaluated for n-iteration. We have

$$\left\|\boldsymbol{B}^*-\boldsymbol{B}^{(n)}\right\|_\nu\leq\frac{1}{1-\theta}\left\|\Delta\boldsymbol{I}^{(n)}\right\|_\nu \tag{7}$$

At each iteration the magnetic field $(\boldsymbol{B}^{(n)},\boldsymbol{H}^{(n)})$ verifies relations (1) and (2). The error is given only by the constitutive relation $\hat{\boldsymbol{F}}$:

$$\left\|\boldsymbol{H}^{(n)}-\hat{\boldsymbol{F}}(\boldsymbol{B}^{(n)})\right\|_\mu=\left\|\Delta\boldsymbol{I}^{(n)}\right\|_\nu \tag{8}$$

The error for H can be evaluated by the relation:

$$\left\|\boldsymbol{H}^*-\boldsymbol{H}^{(n)}\right\|_\mu\leq\frac{1}{1-\theta}\left\|\Delta\boldsymbol{I}^{(n)}\right\|_\nu.$$

In free space Ω_{air} we have:

$$\int_{\Omega_{air}}\nu\left(\boldsymbol{B}^*-\boldsymbol{B}^{(n)}\right)^2d\Omega\leq\frac{\theta}{2\sqrt{1-\theta}}\left\|\Delta\boldsymbol{I}^{(n)}\right\|_\nu$$

This relation is much more convenient in the case of the contraction factors θ very close to unit.

D. The Overrelaxation

The main disadvantage of the polarization is its slow convergent speed. An important improvement can be obtained by overrelaxation. From relations (7), (8), it results that for a smaller value of $\|\Delta\boldsymbol{I}^{(n)}\|_\nu$ the errors becomes smaller. If $\boldsymbol{I}^{(k)}=\hat{\boldsymbol{W}}(\boldsymbol{I}^{(k-1)})$, then $\boldsymbol{I}^{(k)}=\boldsymbol{I}^{(k-1)}+\omega(\boldsymbol{I}^{(k)}-\boldsymbol{I}^{(k-1)})$ is sought so that $h(\omega)=\|\hat{\boldsymbol{W}}(\boldsymbol{I}^{(k)})-\boldsymbol{I}^{(k)}\|_\nu^2$ is as small as possible [4]. The numerical overrelaxation procedure has the following steps:

1)

$$\boldsymbol{B}^{(k)}=\hat{\boldsymbol{B}}\left(\boldsymbol{I}^{(k-1)}\right);$$

2)

$$\boldsymbol{I}^{(k)}=\hat{\boldsymbol{G}}\left(\boldsymbol{B}^{(k)}\right);$$

3)

$$\boldsymbol{B}^{(k+1)} = \hat{\boldsymbol{B}}\left(\boldsymbol{I}^{(k)}\right);$$

4)

$$h(1) = \left\|\hat{\boldsymbol{G}}\left(\boldsymbol{B}^{(k+1)}\right) - \boldsymbol{I}^{(k)}\right\|_{\nu}^{2};$$

and if $h(1) < \varepsilon$ *where* ε *is fixed, we stop the iteration, otherwise*:

5)

$$\Delta\boldsymbol{I}^{(k)} = \boldsymbol{I}^{(k)} - \boldsymbol{I}^{(k+1)},\ \Delta\boldsymbol{B}^{(k+1)} = \boldsymbol{B}^{(k+1)} - \boldsymbol{B}^{(k)}$$

6) *The solving of the equation:*

$$\frac{1}{2}h'(\omega) = \left\langle \left.\frac{d\hat{\boldsymbol{G}}}{d\boldsymbol{B}}\right|_{\boldsymbol{B}^{(k)}+\omega\Delta\boldsymbol{B}^{(k+1)}} \left(\Delta\boldsymbol{B}^{(k+1)} - \Delta\boldsymbol{I}^{(k)}\right),\ \hat{\boldsymbol{G}}\left(\boldsymbol{B}^{(k)} + \omega\Delta\boldsymbol{B}^{(k+1)}\right) - \left(\boldsymbol{I}^{(k-1)} + \omega\Delta\boldsymbol{I}^{(k)}\right)\right\rangle = 0; \quad (9)$$

7) $\boldsymbol{I}^{(k+1)} = \hat{\boldsymbol{G}}(\boldsymbol{B}^{(k)} + \omega\Delta\boldsymbol{B}^{(k+1)})$, *go to 1)*

$\boldsymbol{B}$ and $\boldsymbol{I}$ are numerically defined by the values in a finite number of subdomains. Because the function $\hat{\boldsymbol{G}}$ is generated by the local function $\boldsymbol{g}$, it results that $\hat{\boldsymbol{G}}$ and $(d\hat{\boldsymbol{G}}/d\boldsymbol{B})$ can be easily calculated. Equation (9) is solved by the secant method, a number of 3–5 iterations being enough. Using this overrelaxation procedure, Chiampi [7] reported a significant increase of the convergence speed, especially for the first part of the iteration scheme. The overrelaxation factors ω have values between 1, 2 and 50.

IV. FEM Solving of the Linear Field Problem

For each iteration we have to calculate the magnetic field $(\boldsymbol{B}_T,\ \boldsymbol{H}_T)$ which verifies the equations: $\nabla \times \boldsymbol{H}_T = \boldsymbol{J}$, $\nabla\boldsymbol{B}_T = \rho$, $\boldsymbol{B}_T = \mu\boldsymbol{H}_T + \boldsymbol{I}_T$. We can obtain easily a magnetic field $(\boldsymbol{B}_s, \boldsymbol{H}_s)$ so that $\nabla\times\boldsymbol{H}_s = \boldsymbol{J}$ and $\nabla\boldsymbol{B}_s = \rho$ (For example, using Biot–Savart–Laplace and Coulomb formulae.) Also we can build a magnetic field $(\boldsymbol{B}_A, \boldsymbol{H}_A)$ having $\nabla \times \boldsymbol{H}_A = 0$ and $\nabla\boldsymbol{B}_A = 0$ and the boundary conditions $\boldsymbol{H}_{tA} = \boldsymbol{H}_{tT} - \boldsymbol{H}_{ts}$ and $B_{nA} = B_{nT} - B_{ts}$ (For example using a scalar potential for $\boldsymbol{H}_A$ and a vector potential for $\boldsymbol{B}_A$). The magnetic fields $(\boldsymbol{B}_s, \boldsymbol{H}_s)$ and $(\boldsymbol{B}_A, \boldsymbol{H}_A)$ do not have any constitutive restrictions. It remain to compute the magnetic field $(\boldsymbol{B}, \boldsymbol{H}) = (\boldsymbol{B}_T, \boldsymbol{H}_T) - (\boldsymbol{B}_s, \boldsymbol{H}_s) - (\boldsymbol{B}_A, \boldsymbol{H}_A)$ having zero boundary conditions and zero sources. It verifies the equations:

$$\nabla \times \boldsymbol{H} = \boldsymbol{0}, \qquad \nabla\boldsymbol{B} = 0, \qquad \boldsymbol{B} + (-\mu\boldsymbol{H}) = \boldsymbol{I} \quad (10)$$

where $\boldsymbol{I} = \boldsymbol{I}_T + B_s + \boldsymbol{B}_A - \mu(\boldsymbol{H}_s + \boldsymbol{H}_A)$. Let $\boldsymbol{L}'$ be the space of the fields $\boldsymbol{H}$ which have the boundary condition $\boldsymbol{H}_t = \boldsymbol{0}$ on the surface S'. Let $\boldsymbol{L}''$ be the space of the fields $\boldsymbol{B}$ which have the boundary condition $B_n = 0$ on the surface S''. We have

$$\langle \boldsymbol{B}, \boldsymbol{H}\rangle = \langle \boldsymbol{B}, \mu\boldsymbol{H}\rangle_{\nu} = 0. \quad (11)$$

Hence it results that $\boldsymbol{L}'$, $\boldsymbol{L}''$ are orthogonal in $\boldsymbol{L}^2(\Omega)$ and $\mu\boldsymbol{L}'$, $\boldsymbol{L}''$ are orthogonal in $\boldsymbol{L}^2_{\nu}(\Omega)$. From relation (10) it results that the solving of the magnetic field problem $(\boldsymbol{B}, \boldsymbol{H})$ consists in the decomposition of the polarization $\boldsymbol{I}$ in the spaces $\mu\boldsymbol{L}'$ and $\boldsymbol{L}''$ [8]. From relation (11) it results that the solving of the magnetic field problem $(\boldsymbol{B}, \boldsymbol{H})$ consists in the decomposition of the polarization $\boldsymbol{I}$ in its orthogonal components $\boldsymbol{B}$ and $-\mu\boldsymbol{H}$ (Fig. 1). This is the same with the minimization of the distance $d^2(\boldsymbol{X}) = \|\boldsymbol{I} - \boldsymbol{X}\|^2_{\nu}$, or of the functional

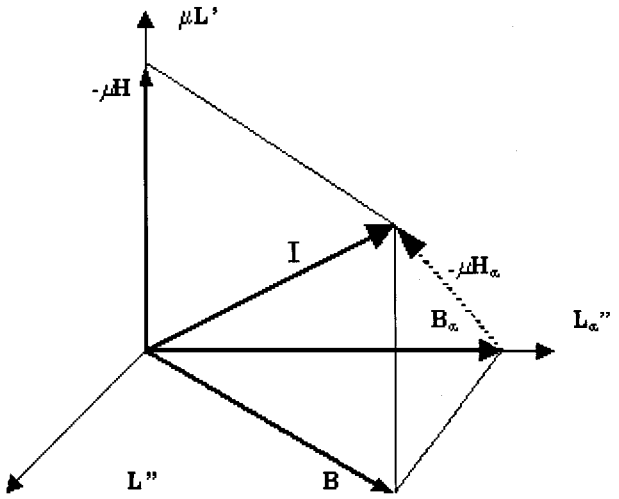

Fig. 1. Decomposition of the polarization

$$F(\boldsymbol{X}) = -2\langle \boldsymbol{I}, \boldsymbol{X}\rangle_{\nu} + \|\boldsymbol{X}\|^2_{\nu} \quad (12)$$

in the space $\boldsymbol{L}''$, when we obtain $\boldsymbol{X} = \boldsymbol{B}$ or in the space $\mu\boldsymbol{L}'$, when we obtain $\boldsymbol{X} = -\mu\boldsymbol{H}$. When $\boldsymbol{X} \in \mu\boldsymbol{L}'$, we can use the formulation in scalar potential $\boldsymbol{X} = -\mu\nabla\Phi$, with the boundary condition $\Phi = 0$ on the surface S'. When $\boldsymbol{X} \in \boldsymbol{L}''$, we can use the formulation in vector potential $\boldsymbol{X} = \nabla \times \boldsymbol{A}$, with the boundary condition $\boldsymbol{A}_t = 0$ on S''.

In FEM the minimization of the functional (12) is done in finite subspace $\boldsymbol{L}''_{\alpha}$ of $\boldsymbol{L}''$ (Fig. 1) or in subspace $\boldsymbol{L}'_{\beta}$ of $\mu\boldsymbol{L}'$. The component $\boldsymbol{B}_{\alpha}$ of the polarization $\boldsymbol{I}$ on the finite subspace $\boldsymbol{L}''_{\alpha}$ is the same as the component of $\boldsymbol{B}$ on this subspace. Therefore the FEM numerical computation of the approximate solution $\boldsymbol{B}_{\alpha}$ results as a composed function $\hat{\boldsymbol{P}}_{\alpha} \circ \hat{\boldsymbol{B}}$ where $\hat{\boldsymbol{P}}_{\alpha}(\boldsymbol{B}) = \boldsymbol{B}_{\alpha}$. FEM numerical scheme of the polarization method is:

$$\cdots \rightarrow \boldsymbol{I}^{(n-1)} \xrightarrow{\hat{\boldsymbol{B}}} \boldsymbol{B}^{(n)} \xrightarrow{\hat{\boldsymbol{P}}_{\alpha}} \boldsymbol{B}^{(n)}_{\alpha} \xrightarrow{\hat{\boldsymbol{G}}} \boldsymbol{I}^{(n)} \rightarrow \cdots$$

Because $\hat{\boldsymbol{P}}_{\alpha}$ and $\hat{\boldsymbol{B}}$ are nonexpansive, the above method leads to the Picard–Banach sequence of the contractive function $\hat{\boldsymbol{G}} \circ \hat{\boldsymbol{P}}_{\alpha} \circ \hat{\boldsymbol{B}}$. It results

Theorem 3: If the polarization $\boldsymbol{I}$ *is corrected by* $\boldsymbol{B}$ *and the functional (12) is minimized in a subspace with restriction* $\nabla\boldsymbol{X} = 0$, *then the numerical approximation of the polarization method is convergent.*

The dual numerical scheme of the $\boldsymbol{H}$-corrected polarization method is

$$\cdots \rightarrow \boldsymbol{M}^{(n-1)} \xrightarrow{\hat{\boldsymbol{H}}} \boldsymbol{H}^{(n)} \xrightarrow{\hat{\boldsymbol{P}}_{\beta}} \boldsymbol{H}^{(n)}_{\beta} \xrightarrow{\hat{\boldsymbol{G}}'} \boldsymbol{M}^{(n)} \rightarrow \cdots$$

where the component $\boldsymbol{H}_{\beta}$ of $\boldsymbol{H}$ on a finite subspace is $\boldsymbol{H}_{\beta} = \hat{\boldsymbol{P}}_{\beta}(\boldsymbol{H})$.

Theorem 4: If the magnetization $\boldsymbol{M}$ *is corrected by* $\boldsymbol{H}$ *and the functional (12) is minimized in a subspace with restriction*

$\nabla \times \boldsymbol{X} = \boldsymbol{0}$, *then the numerical approximation of the polarization method is convergent*.

V. Eddy Current Problems

We consider conducting ferromagnetic bodies Ω_k moving in the air region Ω_0 with a given speed $\boldsymbol{v}$ [9]–[12]. For each iteration, we have to solve the following set of equation:

in Ω_k:

$$\nabla \times \boldsymbol{E}^{(n)} = -\frac{\partial \boldsymbol{B}^{(n)}}{\partial t}; \quad \nabla \times \boldsymbol{H}^{(n)} = \boldsymbol{J}^{(n)};$$
$$\nabla \boldsymbol{B}^{(n)} = 0; \quad \boldsymbol{J}^{(n)} = \sigma \boldsymbol{E}^{(n)};$$
$$\boldsymbol{B}^{(n)} = \mu \boldsymbol{H}^{(n)} + \boldsymbol{I}^{(n-1)};$$

in Ω_0:

$$\nabla \times \boldsymbol{H}^{(n)} = \boldsymbol{J}_0; \quad \nabla \boldsymbol{B}^{(n)} = 0; \quad \boldsymbol{B}^{(n)} = \mu_0 \boldsymbol{H}^{(n)},$$

where the current density $\boldsymbol{J}_0$ is imposed. The equations are written in the local frame of the bodies. For any $\boldsymbol{I}$ we have one and only one field $\boldsymbol{B}$ that verifies the above equations.

The function $\boldsymbol{I} \to \boldsymbol{B} = \hat{\boldsymbol{B}}(\boldsymbol{I})$ is nonexpansive [12]:

$$\left\| \hat{\boldsymbol{B}}(\boldsymbol{I}') - \hat{\boldsymbol{B}}(\boldsymbol{I}'') \right\|_\nu \le \left\| \boldsymbol{I}' - \boldsymbol{I}'' \right\|_\nu,$$

where

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \int_0^t \int_\Omega \boldsymbol{u}\boldsymbol{v}\, d\Omega\, d\tau \quad \text{and} \quad \langle \boldsymbol{u}, \boldsymbol{v} \rangle_\nu = \langle \boldsymbol{u}, \nu \boldsymbol{v} \rangle.$$

VI. Other Applications of the Picard–Banach Scheme

A. Nonlinear E–J Relation

For nonlinear conducting bodies we replace the nonlinear relation $\boldsymbol{J} = \hat{\boldsymbol{F}}(\boldsymbol{E})$ by $\boldsymbol{E} = \rho \boldsymbol{J} + \boldsymbol{S}$ where $\boldsymbol{S} = \boldsymbol{E} - \rho \hat{\boldsymbol{F}}(\boldsymbol{E}) = \hat{\boldsymbol{G}}(\boldsymbol{E})$. For each iteration we have to solve the electromagnetic field equation: $\nabla \times \boldsymbol{E}^{(n)} = -(\partial \boldsymbol{B}^{(n)}/\partial t)$; $\nabla \times \boldsymbol{H}^{(n)} = \boldsymbol{J}^{(n)} + (\partial \boldsymbol{D}^{(n)}/\partial t)$; $\boldsymbol{D}^{(n)} = \varepsilon \boldsymbol{E}^{(n)}$; $\boldsymbol{H}^{(n)} = \nu \boldsymbol{B}^{(n)}$; $\boldsymbol{E}^{(n)} = \rho \boldsymbol{J}^{(n)} + \boldsymbol{S}^{(n-1)}$. For any $\boldsymbol{S}$ we have one and only one field $\boldsymbol{E}$ which verifies the above equations. It can be proved that the function $\boldsymbol{S} \to \boldsymbol{E} = \hat{\boldsymbol{E}}(\boldsymbol{E})$ is nonexpansive. The field of the imposed sources $\boldsymbol{S}$ is iteratively corrected by $\boldsymbol{E}$: $\boldsymbol{S}^{(n)} = \hat{\boldsymbol{G}}(\boldsymbol{E}^{(n)})$. If the function $\hat{\boldsymbol{F}}$ is Lipschitzian and uniform monotone, then we can choose ρ so that the function $\hat{\boldsymbol{G}}$ is a contraction.

Remark: It can be proved that even if the function $\hat{\boldsymbol{F}}$ is only Lipschitzian we can choose ρ and the time interval $[0, t]$ so that the composed function $\hat{\boldsymbol{G}} \circ \hat{\boldsymbol{E}}$ (not $\hat{\boldsymbol{E}}$) is a contraction.

B. Electric Circuits with Nonlinear Resistors

The nonlinear current–voltage relations $\boldsymbol{y} = \hat{\boldsymbol{f}}(\boldsymbol{x})$ can be replaced by $\boldsymbol{x} = \alpha \boldsymbol{y} + \boldsymbol{s}$ where $\boldsymbol{s} = \boldsymbol{x} - \alpha \hat{\boldsymbol{f}}(\boldsymbol{x}) = \hat{\boldsymbol{g}}(\boldsymbol{x})$. For given sources $\boldsymbol{s}$ we solve the circuit and after we correct the sources by the function $\hat{\boldsymbol{g}}$ [13]. For Lipschitzian and uniform monotone function $\hat{\boldsymbol{f}}$, the reasoning is similar to those for the magnetic static fields. Sometimes, for resistive circuits, we can have only Lipschitzian resistors. For circuits in the transient regime also we can have only Lipschitzian resistors if the voltage controlled ports are connected in parallel with a capacitor and the current controlled ports are connected in series with an inductor.

VII. Conclusions

The polarization method is a Picard–Banach fixed point procedure. Nonlinear medium is replaced with a linear one, where the polarization is corrected by the flux density or magnetic field. So the magnetic permeability is the same for each iteration and therefore the matrix of the equation system may be computed only one time, before the beginning of the iterations. If a homogeneous medium is chosen, then the Green function may be used [for example $(1/4\pi r)$ for unbounded space R^3]. The convergence speed may be increased by an overrelaxation scheme. The errors in comparison with exact solution may be easily evaluated. The FEM solving of the linear problem keeps the convergence of the polarization method.

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