Some mathematical considerations

This paragraph reviews some concepts and mathematical relationships absolutely indispensably for a substantial approach of the electromagnetic field theory. These are going to be briefly presented, assuming they are also known form previously studied disciplines (algebra, vector analysis, differential and integral computation, special mathematics).

● Three-orthogonal coordinate systems

A three-orthogonal coordinate system is a reference system defined by three surface families (marked in a well-defined order by the scalar quantities u_1, u_2 and u_3 called coordinated) which intersect orthogonally. Next, we recall the expressions for the line elements dl_1, dl_2 and dl_3 (their unit tangents being $\bar{e}_1$, $\bar{e}_2$ and $\bar{e}_3$), area elements $dA_1 = ds_2 \cdot ds_3$, $dA_2 = ds_3 \cdot ds_1$ and $dA_3 = ds_1 \cdot ds_2$ and volume elements $dv = ds_1 \cdot ds_2 \cdot ds_3$ corresponding to the Cartesian, cylindrical and spherical coordinate systems.

For Cartesian coordinate system:

$u_1 = x$, $u_2 = y$, $u_3 = z$, $\bar{e}_1 = \bar{e}_x = \bar{i}$, $\bar{e}_2 = \bar{e}_y = \bar{j}$, $\bar{e}_3 = \bar{e}_z = \bar{k}$,

$dl_1 = dx$, $dl_2 = dy$, $dl_3 = dz$,

$dA_1 = dy \cdot dz$, $dA_2 = dz \cdot dx$, $dA_3 = dx \cdot dy$, $dv = dx \cdot dy \cdot dz$.

For cylindrical coordinate system:

$u_1 = z$, $u_2 = r$, $u_3 = \varphi$, $\bar{e}_1 = \bar{e}_z$, $\bar{e}_2 = \bar{e}_r$, $\bar{e}_3 = \bar{e}_\varphi$,

$dl_1 = dz$, $dl_2 = dr$, $dl_3 = r \cdot d\varphi$,

$dA_1 = r \cdot dr \cdot d\varphi$, $dA_2 = r \cdot dz \cdot d\varphi$, $dA_3 = dz \cdot dr$, $dv = r \cdot dz \cdot dr \cdot d\varphi$.

For spherical coordinate system:

$u_1 = r, u_2 = \theta$, $u_3 = \varphi$, $\overline{e_1} = \overline{e_r}, \overline{e_2} = \overline{e_\theta}$, $\overline{e_3} = \overline{e_\varphi}$,

$dA_1 = r^2 \cdot \sin\theta \cdot d\theta \cdot d\varphi, dA_2 = r \cdot \sin\theta \cdot dr \cdot d\varphi, dA_3 = r \cdot dr \cdot d\theta$,
$dv = r^2 \cdot \sin\theta \cdot dr \cdot d\theta \cdot d\varphi$.

● Dot (scalar) product, cross (vector) product

Let two vectors, denoted $\overline{a}$ and $\overline{b}$, having the absolute value a and respectively, b, which define the surface (Π), and let be γ the angle between $\overline{a}$ and $\overline{b}$, with $\gamma \leq \pi\,[\text{rad}]$.

By dot product of vectors $\overline{a}$ and $\overline{b}$ one understands the vector

$$\overline{a} \cdot \overline{b} = a \cdot b \cdot \cos\,\gamma\,.$$

By cross product of vectors $\overline{a}$ and $\overline{b}$ one understands the vector

$$\overline{a} \times \overline{b} = (a \cdot b \cdot \sin\,\gamma) \cdot \overline{n},$$

having the absolute value $a \cdot b \cdot \sin\,\gamma$, the unit vector $\overline{n}$ being perpendicular on the surface (Π) and having the orientation given by the corkscrew rule (corresponding to the rotation in this plane of vector $\overline{a}$ in order to superpose it over vector $\overline{b}$ following the shortest path).

The following properties are underlined $\overline{a} \cdot \overline{b} = \overline{b} \cdot \overline{a}$ and $\overline{a} \times \overline{b} = -(\overline{b} \times \overline{a})$.

● A few definitions related to vector fields

Let a vector field $\overline{m_v}$. A **field line** of this vector field is an oriented spatial curve, having the property that, in any of its points, the vector

$$\overline{m_v} = m_{v1} \cdot \overline{e_1} + m_{v2} \cdot \overline{e_2} + m_{v3} \cdot \overline{e_3}$$

is tangent to it and it is oriented the same way the curve is (see Figure 1.2.1). m_{v1}, m_{v2} and m_{v3} are the scalar components of the vector $\overline{m}_v$.

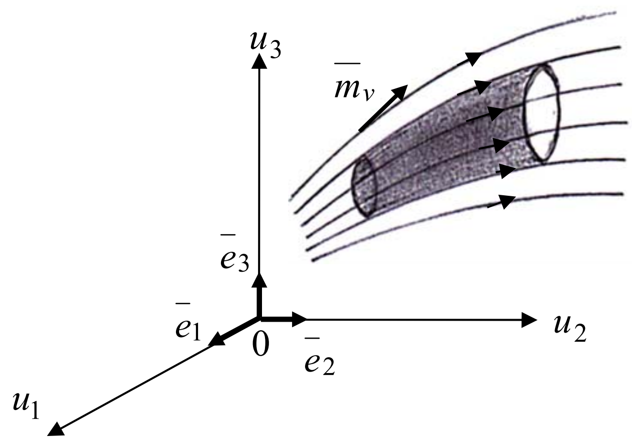

Figure 1.2.1

The field line spectrum is a field line assembly from a certain area in the space. The **flux tube** is a tubular surface defined by the totality of the field lines which pass through all the points belonging to a small closed contour (see the hachured area from Figure 1.2.1).

● Integrals

Line integral of a vector field

Let consider a vector field $\overline{m}_v$ and an open space curve (C) arbitrary oriented by the unit tangent vector $\overline{t}$. One realizes a discretization of the curve domain (C) in elementary line sub-domains (centered respectively on points P_k where the vector field has the value $\overline{m}_{v,k}$) of lengths Δl_k and there are defined the elementary vectors $\overline{\Delta l_k}$ using the relation $\overline{\Delta l_k} = \Delta l_k \cdot \overline{t}$ (see Figure 1.2.2).

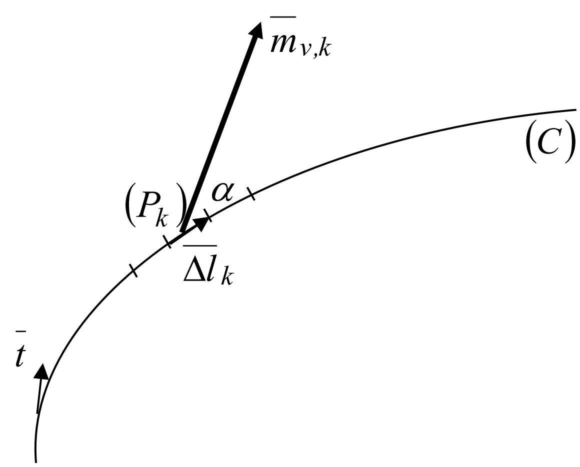

Figure 1.2.2

One computes the sum $\sum_{k=1}^{n} \overline{m}_{v,k} \cdot \overline{\Delta l}_k$, sum which becomes limit for $Max\{\Delta l_k\} \to 0$ (equivalent to $n \to \infty$). If the limit exists, then

$$\Im_c = \lim_{Max\{\Delta l_k \to 0\}} \sum_{k=1}^{n} \overline{m}_{v,k} \cdot \overline{\Delta l}_k = \int_{(C)} \overline{m}_v \cdot \overline{dl} = \int_{(C)} m_v \cdot dl \cdot \cos\alpha = \int_{(C)} m_{v,tg} \cdot dl = \tilde{m}_{v,tg} \cdot Lg(C)$$ is

called **the line integral** of the vector field $\overline{m}_v$ along the oriented curve (C). With $\tilde{m}_{v,tg}$ was denoted the average value of the scalar tangential component of the vector field $\overline{m}_v$ along the curve (C), and $Lg(C)$ is the length of curve (C).

Surface integral of a vector field

Let consider a vector field $\overline{m}_v$ and an open surface (S) arbitrary oriented by the normal unit vector $\overline{n}$. One realizes a discretization of the surface domain (S) in elementary sub-domains (centered respectively on points P_k, where the vector field has the value $\overline{m}_{v,k}$) of areas ΔA_k and there are defined the elementary vectors $\overline{\Delta A}_k$ by relation $\overline{\Delta A}_k = \Delta A_k \cdot \overline{n}$ (see Figure 1.2.2). One computes the sum $\sum_{k=1}^{n} \overline{m}_{v,k} \cdot \overline{\Delta A}_k$, sum which becomes limit for $Max\{\Delta A_k\} \to 0$ (equivalent to $n \to \infty$).

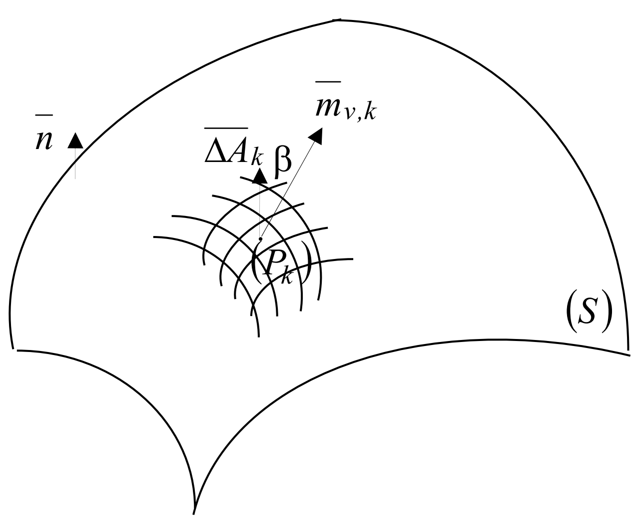

Figure 1.2.3

If this limit exists, then

$$\Im_s = \lim_{Max\{\Delta A_k \to 0\}} \sum_{k=1}^{n} \overline{m}_{v,k} \cdot \overline{\Delta A}_k = \int_{(C)} \overline{m}_v \cdot \overline{dA} = \int_{(S)} m_v \cdot dA \cdot \cos\beta =$$

$$= \int_{(S)} m_{v,nm} \cdot dl = \tilde{m}_{v,nm} \cdot Aria(S)$$

is called **the surface integral** of the vector field $\overline{m}_v$ along oriented surface (S). With $\tilde{m}_{v,nm}$ was denoted the average value of the scalar normal component of the vector field $\overline{m}_v$ along the surface (S), and $Aria(S)$ is the aria of the surface (S).

Volume integral of a scalar field

Let consider a scalar field m_s and a volume domain (V). One realizes a discretization of the volume domain (V) in elementary sub-domains (centered respectively on points P_k, where the vector field has the value $m_{s,k}$), of volumes Δv_k (see Figure 1.2.4). One computes the sum $\sum_{k=1}^{n} m_{s,k} \cdot \Delta v_k$, sum which becomes a limit for $Max\{\Delta v_k\} \to 0$ (equivalent to $n \to \infty$). If the limit exists, then

$$\Im_v = \lim_{Max\{\Delta v_k \to 0\}} \sum_{k=1}^{n} m_{s,k} \cdot \Delta v_k = \int_{(V)} m_s \cdot dv = \tilde{m}_s \cdot Vol(V)$$

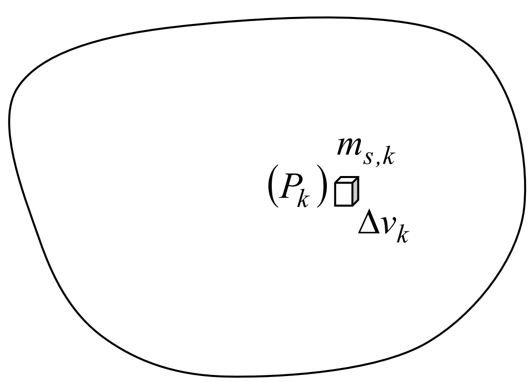

Figure 1.2.4

is called **the volume integral** of the scalar field m_s on domain (V). With $\tilde{m}_s$ was denoted the mean value of the scalar field m_s on volume $Vol(V)$.

Remarks

1. In order to define the integrals using line vectors and surface vectors associated to some closed curves (Γ), respectively to some closed surfaces (Σ), the computation is similar to the one presented above for open curves and, respectively, for open surfaces.

2. Whenever we deal with a closed surface (Σ), **by convention we choose the orientation of the normal unit vector** $\overline{n}$ towards the **exterior of that surface**.

3. The results of the integrals presented above are scalars affected by sign, which represent some global quantities associated to line, surface and volume domains, quantities which give information about **the average** behavior of the integrands-local quantities on the domains on which these integrals were computed.

4. In time-varying state the integrands are local quantities depending on point and time, while the integrals are global quantities which depend only on time.

5. The orientation of a curve by optional choice of the sense for the unit vector $\overline{t}$, respectively the orientation of a surface by optional choice of the sense of the unit vector $\overline{n}$, represents the choice of a **reference sense** for the line integral computation, respectively for the surface integral computation. The significance of the sign of the integral is that the **real sense** of the global quantity coincides (if the integral is positive) or does not coincide (if the integral is negative) with the reference sense arbitrary chosen for the orientation of the line element vector $\overline{\Delta l}$, respectively area element vector $\overline{\Delta A}$.

6. For numerical computation, the integrals are being approximated by the sums from which they are derived, and the finer the discretization is, the closer the results are to the exact solutions.

● Space differential operators

The differential operators represent extremely important mathematical instruments when characterizing the local behavior of scalar and vector fields.

From these, only six are going to be presented, three of them referring to continuity domains and three to discontinuity surfaces.

The divergence is a differential operator applicable to **vectors** and it has as a result a **scalar**.

The curl is a differential operator applicable to **vectors** and it has as a result a **vector.**

The gradient is a differential operator applicable to **scalars** and it has as a result a **vector**.

We present the expressions of these operators for continuity domains, in Cartesian coordinates. If

$$\overline{m}_v = \overline{m}_v(x,y,z,t) = m_{vx}(x,y,z,t)\cdot\bar{i} + m_{vy}(x,y,z,t)\cdot\bar{j} + m_{vz}(x,y,z,t)\cdot\bar{k},$$

and

$$m_s = m_s(x,y,z,t),$$

then

$$div\,\overline{m}_v = \frac{\partial m_{vx}(x,y,z,t)}{\partial x} + \frac{\partial m_{vy}(x,y,z,t)}{\partial y} + \frac{\partial m_{vz}(x,y,z,t)}{\partial z},$$

$$curl\,\overline{m}_v = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ m_{vx}(x,y,z,t) & m_{vy}(x,y,z,t) & m_{vz}(x,y,z,t) \end{vmatrix},$$

$$grad\, m_s(x,y,z,t) = \frac{\partial m_s(x,y,z,t)}{\partial x}\cdot \overline{i} + \frac{\partial m_s(x,y,z,t)}{\partial y}\cdot \overline{j} + \frac{\partial m_s(x,y,z,t)}{\partial z}\cdot \overline{k}\,.$$

There are a lot of field problems referring to discontinuity domains.

Let a surface which separates two media with different properties (denoted by (1) and (2) in Figure 1.2.5), and let $\overline{n}_{12}$ be its normal unit vector in the point in which

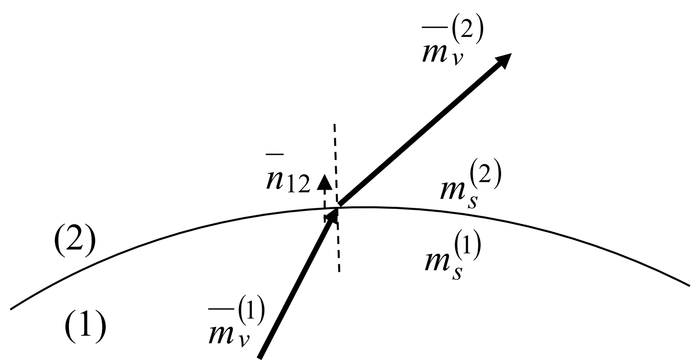

Figure 1.2.5

we wish to use the differential operators (oriented from medium (1) towards medium (2)) and let $\overline{m}_v^{(1)}$ and $\overline{m}_v^{(2)}$, respectively $m_s^{(1)}$ and $m_s^{(2)}$, the values of the local quantities $\overline{m}_v$ and, respectively, m_s in the nearby neighborhood of the separation surface between the two media. One defines the following surface operators:

▫ **the surface divergence**

$$div_s\, \overline{m}_v = \overline{n}_{12} \cdot \left(\overline{m}_v^{(2)} - \overline{m}_v^{(1)} \right),$$

▫ **the surface curl**

$$curl_s\, \overline{m}_v = \overline{n}_{12} \times \left(\overline{m}_v^{(2)} - \overline{m}_v^{(1)} \right),$$

▫ **the surface gradient**

$$grad_s\, m_s = \overline{n}_{12} \cdot \left(m_s^{(2)} - m_s^{(1)} \right).$$

Remarks

1. To zero divergence vector fields correspond closed field lines.

2. To zero curl vectors fields correspond open field lines.

● Integral relationships

Gauss – Ostrogradski relationships expresses the flux of a vector field $\overline{m}_v$ through an open surface (Σ) by a volume integral of the divergence of the field $\overline{m}_v$ on domain (V_Σ) bounded by the closed surface (Σ):

$$\int_{(\Sigma)} \overline{m}_v \cdot \overline{dA} = \int_{(V_\Sigma)} \left(div\, \overline{m}_v \right) \cdot dv .$$

Stokes relationships expression expresses the circulation of a vector field $\overline{m}_v$ along a closed curve (Γ) by a surface integral of the curl of the field $\overline{m}_v$ on an open surface (S_Γ) bounded by the closed curve (Γ):

$$\int_{(\Gamma)} \overline{m}_v \cdot \overline{dl} = \int_{(S_\Gamma)} \left(curl\, \overline{m}_v \right) \cdot \overline{dA} .$$

● Time derivative of a space integral

As we know, the time evolution of a physical quantity is expressed from mathematical point of view by its time derivative. The problem becomes more complicated if we want to study the time evolution of a global quantity (expressed by an integral) if we deal with moving bodies, when the global quantity varies in time both due to the time integrand variation (in the present case a vector field $\overline{m_v}$ or a scalar one m_s), and because of the time evolution of the integration domain. In this situation the integration domains are driven by moving bodies and the time derivatives of the space and volume integrals are computed using the relations:

$$\frac{d}{dt}\int_{(S_\Gamma)} \overline{m_v}\cdot\overline{dA} = \int_{(S_\Gamma)}\left(\frac{\partial\overline{m_v}}{\partial t} + w\cdot div\,\overline{m_v} + curl\left(\overline{m_v}\times\overline{w}\right)\right)\cdot\overline{dA},$$

$$\frac{d}{dt}\int_{(V_\Sigma)} m_s\cdot dv = \int_{(V_\Sigma)}\left(\frac{\partial m_s}{\partial t} + div\left(\overline{w\cdot m_s}\right)\right)\cdot dv,$$

where $\overline{w}$ is the local speed vector.